


Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year		Final Term Exam Date: January 10 , 2017 Course: Mathematics 1 – A Duration: 3 hours
<ul style="list-style-type: none"> <li>• The Exam consists of one page</li> <li>• Answer <b>All</b> Questions</li> </ul>	<ul style="list-style-type: none"> <li>• No. of questions: 4</li> <li>• Total Mark: 100</li> </ul>	
<b><u>Question 1</u></b>		
(a) Find $y'$ from the following:		12
(i) $y = x^{-3} + 4x^2 - 2$ (ii) $y = \sin x^2 \cdot \sec x$ (iii) $y = \ln \ln x + \cos \ln x$ (iv) $y = \sec^3 x + \log x$ (v) $y^3 + e^{xy} + x^x = 0$ (vi) $y = t \tan t, x = t \ln t$		
(b) Find the following limits:		
(i) $\lim_{x \rightarrow \pi} \frac{\tan 2x}{\sin x}$ (ii) $\lim_{x \rightarrow 0} \frac{\ln(1 + 2x)}{3^x - 4^x}$ (iii) $\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + \ln x}$ (iv) $\lim_{x \rightarrow \infty} \frac{x^6 - 3^x}{x^5 + 4^x}$		8
<b><u>Question 2</u></b>		
(a) Write the Maclurin's expansion of the function: $f(x) = x \cos x^2$ .		4
(b) State and discuss Rolle's theorem for $f(x) = \sqrt[3]{(x-1)^2}$ in interval $[0, 2]$ .		5
(c) Sketch the curve of each function : $f(x) = x - \ln x$ , $g(x) = \frac{1}{\sqrt{x^2-1}}$ .		12
(d) Find the integrals: (i) $\int (\frac{1}{x^3} + \frac{1}{3x}) dx$ (ii) $\int (3^x - 4^x)^2 dx$ (iii) $\int \frac{\cos t}{\sqrt{\sin t}} dt$		9
<b><u>Question 3</u></b>		
(a) Find the sum to n terms from the series : $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots$		5
Also, show that its sum to infinity is 0.25 .		
(b) Prove that : $(1 + 2 + 3 + \dots + n)^2 = 1^3 + 2^3 + 3^3 + \dots + n^3$ , $n \in N$ .		5
(c) Sum the series : $1 + x(1 + x) + x^2(1 + x + x^2) \dots$ to n terms.		5
(d) Find the coefficient of $x^{15}$ in the expansion : $(1 + x)^4(1 - x)^{-4}$ .		5
<b><u>Question 4</u></b>		
(a) Resolve the fraction : $\frac{6x^2-8x+3}{(1-x)^3}$ into its partial fractions.		6
(b) Solve the equation : $z^2 = 1 + i\sqrt{3}$ .		6
(c) Discuss the existence of the solution of the linear system :		
$x - 2y + 3z = -2, -x + y - 2z = 3, 2x - y + 3z = 1$		6
(d) Solve the equation $x^4 - 7x^3 + 21x^2 + kx + 30 = 0$ given that $1 + 2i$ is a root. Also, find the value of k .		6
(e) Find the eigenvalues and eigenvectors of the matrix : $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$		6

## *Model Answer*

### Answer of Question 1

(a)(i)  $y' = -3x^{-4} + 4x^2 \cdot \ln 4 \cdot 2x - 0$

(ii)  $y' = \sin x^2 \cdot \sec x \cdot \tan x + \cos x^2 \cdot 2x \cdot \sec x$

(iii)  $y' = \frac{1}{\ln x} \cdot \frac{1}{x} - \sin \ln x \cdot \frac{1}{x}$

(iv)  $y' = 3 \sec^3 x \cdot \tan x + \frac{1}{\ln 10} \cdot \frac{1}{x}$

(v)  $3y^2 \cdot y' + e^{xy}(y + xy') + e^{x \ln x}(1 + \ln x) = 0$

(vi)  $y' = \frac{t \cdot \sec^2 t + \tan t}{1 + \ln t}$

-----12-Marks

(b)(i)  $\lim_{x \rightarrow \pi} \frac{\tan 2x}{\sin x} = \frac{0}{0} = \lim_{x \rightarrow \pi} \frac{2 \sec^2 2x}{\cos x} = -2$

(ii)  $\lim_{x \rightarrow 0} \frac{\ln(1 + 2x)}{3^x - 4^x} = \frac{0}{0} = \frac{2}{\ln(3/4)}$

(iii)  $\lim_{x \rightarrow \infty} \frac{x^2}{x^2 + \ln x} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{2x}{2x + \frac{1}{x}} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{4x}{4x} = 1$

(iv)  $\lim_{x \rightarrow \infty} \frac{x^6 - 3^x}{x^5 + 4^x} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{\frac{x^6}{4^x} - \frac{3^x}{4^x}}{\frac{x^5}{4^x} + 1} = 0$

-----8-Marks

### Answer of Question 2

(a)  $f(x) = x \cos x^2 = x \left[ 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \dots \right] = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \dots$

-----4-Marks

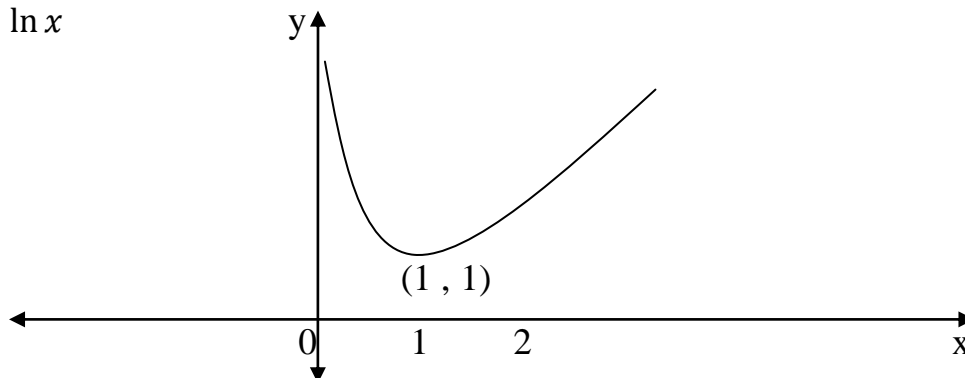
(b) Rolle's theorem.

We see that  $f(x)$  continuous in the given interval  $[0, 2]$  but its derivative  $f'(x) = \frac{2}{3\sqrt[3]{x-1}}$

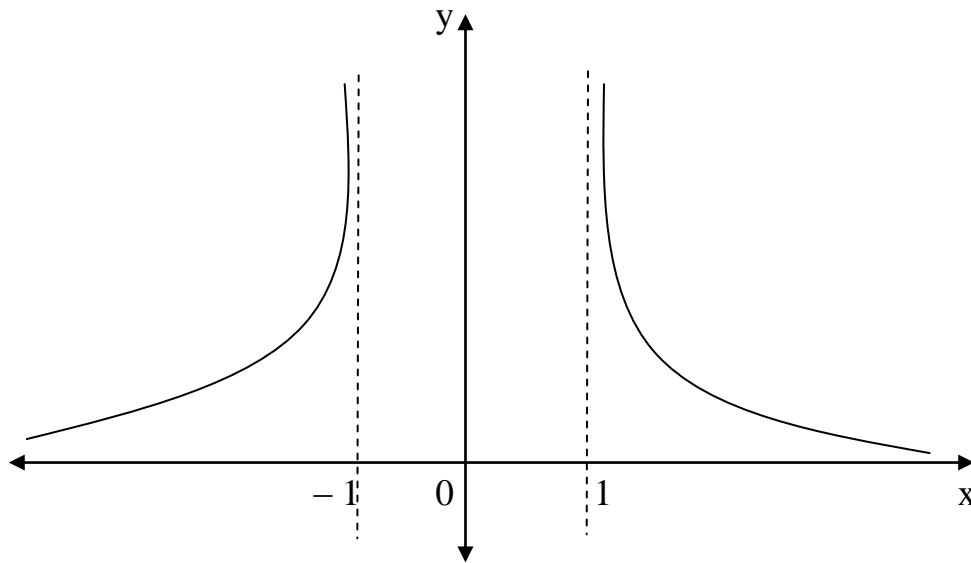
is infinity at 1 and  $f(0) = f(2) = 1$ . The theorem is not satisfied.

-----4-Marks

(c)  $f(x) = x - \ln x$



$$g(x) = \frac{1}{\sqrt{x^2-1}}.$$



-----12-Marks

$$(d)(i) \int \left( \frac{1}{x^3} + \frac{1}{3^x} \right) dx = -\frac{1}{2}x^{-2} + \frac{\left(\frac{1}{3}\right)^x}{\ln(1/3)} + c$$

$$(ii) \int (3^x - 4^x)^2 dx = \int (9^x + 16^x - 2 \cdot 12^x) dx = \frac{9^x}{\ln 9} + \frac{16^x}{\ln 16} - 2 \frac{12^x}{\ln 12} + c$$

$$(iii) \int \frac{\cos t}{\sqrt{\sin t}} dt = \int \cos t (\sin t)^{-1/2} dt = 2 \sqrt{\sin t} + c$$

-----9-Marks

**Dr. Mohamed Eid**

### Answer of Question 3

(a) Given the series  $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots$  find its sum to  $n$  terms. Show that its sum to infinity is 0.25

**Solution**

$$u_r = \frac{1}{r(r+1)(r+2)}$$

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2)}{r(r+1)(r+2)} - \frac{r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)} = 2u_r$$

$$u_r = \frac{1}{2} \left[ \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \right] = \frac{1}{2} [f(r) - f(r+1)]$$

$$S_n = \frac{1}{2} [f(1) - f(n+1)] = \frac{1}{2} \left[ \frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$S_\infty = \lim_{n \rightarrow \infty} \frac{1}{2} \left[ \frac{1}{2} - \frac{1}{(n+1)(n+2)} \right] = \frac{1}{4}$$

---

(b) Prove that  $(1 + 2 + 3 + \dots + n)^2 = 1^3 + 2^3 + 3^3 + \dots + n^3$  for every integer  $n \in \mathbb{N}$ .

**Solution**

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \sum_{r=1}^n r^3 = \sum_{r=1}^n r(r+1)(r+2) - 3r(r+1) + r$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} - \frac{3n(n+1)(n+2)}{3} + \frac{r(r+1)}{2}$$

$$= \frac{3n(n+1)(n+2)(n+3)}{12} - \frac{12n(n+1)(n+2)}{12} + \frac{6n(n+1)}{12}$$

$$= \frac{n(n+1)}{12} (3(n+2)(n+3) - 12(n+2) + 6)$$

$$= \frac{n(n+1)}{12} (3n^2 + 15n + 18 - 12n - 24 + 6)$$

$$= \frac{n(n+1)}{12} (3n^2 + 3n) = \frac{n^2(n+1)^2}{4} = \left[ \frac{n(n+1)}{2} \right]^2$$

$$= \left[ \sum_{r=1}^n r \right]^2 = (1 + 2 + 3 + \dots + n)^2$$

---

(c) Sum the series  $1 + x(1+x) + x^2(1+x+x^2) + \dots$  to  $n$  terms

**Solution**

$$\begin{aligned}
& 1 + x(1+x) + x^2(1+x+x^2) + \dots \text{to } n \text{ terms} \\
& = 1 + x(1+x) + x^2(1+x+x^2) + \dots + x^{n-1}(1+x+x^2+\dots+x^{n-1}) \\
& = 1 + x \frac{(1-x^2)}{1-x} + x^2 \frac{(1-x^3)}{1-x} + \dots + x^{n-1} \frac{(1-x^n)}{1-x} \\
& = \frac{1}{1-x} \left[ 1-x + x(1-x^2) + x^2(1-x^3) + \dots + x^{n-1}(1-x^n) \right] \\
& = \frac{1}{1-x} \left[ (1+x+x^2+\dots+x^{n-1}) - (x+x^3+x^5+\dots+x^{2n-1}) \right] \\
& = \frac{1}{1-x} \left[ \frac{1-x^n}{1-x} - \frac{x(1-x^{2n})}{1-x^2} \right] = \boxed{\frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}}
\end{aligned}$$


---

(d) Find coefficient  $x^{15}$  in the expansion  $(1+x)^4(1-x)^{-4}$ .

**Solution**

$$\begin{aligned}
(1+x)^4(1-x)^{-4} & = (1+4x+6x^2+4x^3+x^4) \left( \sum_{r=0}^{\infty} {}^{-4}C_r (-x)^r \right) \\
& = (1+4x+6x^2+4x^3+x^4) \left( \sum_{r=0}^{\infty} (-1)^r C_r^{4+r-1} (-1)^r x^r \right) \\
& = (1+4x+6x^2+4x^3+x^4) \left( \sum_{r=0}^{\infty} C_r^{3+r} x^r \right) \\
& = (1+4x+6x^2+4x^3+x^4) (C_0^3 + C_1^4 x + C_2^5 x^2 + \dots + C_n^{3+n} x^n + \dots)
\end{aligned}$$

Coefficient  $x^{15}$  is

$$\begin{aligned}
& C_{15}^{18} + 4C_{14}^{17} + 6C_{13}^{16} + 4C_{12}^{15} + C_{11}^{14} \\
& = C_3^{18} + 4C_3^{17} + 6C_3^{16} + 4C_3^{15} + C_3^{14} \\
& = \frac{18 \cdot 17 \cdot 16}{3 \cdot 2 \cdot 1} + 4 \cdot \frac{17 \cdot 16 \cdot 15}{3 \cdot 2 \cdot 1} + 6 \cdot \frac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1} + 4 \cdot \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} + \frac{14 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 1} = 9080
\end{aligned}$$


---

### Answer of Question 4

(a) Resolve  $\frac{6x^2 - 8x + 3}{(1-x)^3}$  into its partial fraction.

**Answer**

$$\frac{6x^2 - 8x + 3}{(1-x)^3} = \frac{A}{(1-x)^3} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)}$$

$$\therefore 6x^2 - 8x + 3 = A + B(1-x) + C(1-x)^2 \quad \text{at } x=1 \quad \text{then } A=1$$

$$\text{compare coefficient } x^2 \quad 6 = C$$

$$\text{compare coefficient } x \quad -8 = -B - 2C \quad \text{then } B = -4$$

$$\frac{6x^2 - 8x + 3}{(1-x)^3} = \frac{1}{(1-x)^3} + \frac{-4}{(1-x)^2} + \frac{6}{(1-x)}$$


---

(b) Solve the equation  $z^2 = 1 + i\sqrt{3}$

*Solution*

The solution of the equation  $z^2 = 1 + i\sqrt{3}$  is  $z = (1 + i\sqrt{3})^{1/2}$  first we put the number  $1 + i\sqrt{3}$  in polar form  $r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$  and  $\cos \theta = \frac{1}{2}, \sin \theta = \frac{\sqrt{3}}{2}$  which show

that  $\theta = \frac{\pi}{3}$  then  $1 + i\sqrt{3} = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$  hence

$$\begin{aligned} z &= (1 + i\sqrt{3})^{1/2} = \sqrt{2} \left[ \cos\left(\frac{2\pi}{3} + 2k\pi\right) + i \sin\left(\frac{2\pi}{3} + 2k\pi\right) \right]^{1/2} \\ &= \sqrt{2} \left[ \cos\left(\frac{2\pi+6k\pi}{3}\right) + i \sin\left(\frac{2\pi+6k\pi}{3}\right) \right]^{1/2} \\ &= \sqrt{2} \left[ \cos\left(\frac{2\pi+6k\pi}{6}\right) + i \sin\left(\frac{2\pi+6k\pi}{6}\right) \right], \quad k = 0, 1 \end{aligned}$$

At  $k=0$

$$z_1 = \sqrt{2} \left[ \cos\left(\frac{2\pi}{6}\right) + i \sin\left(\frac{2\pi}{6}\right) \right]$$

At  $k=1$

$$z_2 = \sqrt{2} \left[ \cos\left(\frac{2\pi+6\pi}{6}\right) + i \sin\left(\frac{2\pi+6\pi}{6}\right) \right] = \sqrt{2} \left[ \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right]$$


---

(c) Discuss the existence of the solution for the system

$$x - 2y + 3z = -2$$

$$-x + y - 2z = 3$$

$$2x - z + 3z = 1$$

*Solution*

First write down the augmented matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix}$$

We won't put down as many words in working this example. Here's the work for this augmented Matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix} \xrightarrow{\substack{R_1+R_2 \rightarrow R_2 \\ -2R_1+R_3 \rightarrow R_3}} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{pmatrix}$$

$$\xrightarrow{-2R_2} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{pmatrix} \xrightarrow{-3R_2+R_3 \rightarrow R_3} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

We won't go any farther in this example. Let's go back to equations to see why

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix} \Rightarrow \begin{array}{l} x - 2y + 3z = -2 \\ y - z = -1 \\ 0 = 8 \end{array}$$

rank  $A <$  rank  $G$  then the system inconsistent (no solution).

(d) Solve the equation  $x^4 - 7x^3 + 21x^2 + kx + 30 = 0$  Given that  $1 + 2i$  is a root of the equation, find the values  $k$ .

**Solution**

Since  $w = 1 + 2i$  is a root then  $\bar{w} = 1 - 2i$  is also root

Let the others roots are  $a$  and  $b$

$$(1 + 2i) + (1 - 2i) + a + b = 7$$

$$2 + a + b = 7$$

$$a + b = 5 \tag{1}$$

$$ab(1 + 2i)(1 - 2i) = 30$$

$$5ab = 30$$

$$ab = 6 \tag{4}$$

From (1) and (2)  $a$  and  $b$  are the roots of the equation  $y^2 - 5y + 6 = 0$  then

$$a = 2 \quad b = 3$$

To find  $k$

$$f(2) = (2)^4 - 7(2)^3 + 21(2)^2 + 2k + 30 = 0$$

$$2k + 74 = 0$$

$$\boxed{k = -37}$$

---

(e) Find the eigen values and eigen vectors for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

**Solution:**

Let the eigen vector in the form  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  we solve the equation  $AX = \lambda X$  i.e.

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

which gives the system of equations

$$x + 2y = \lambda x \quad \text{or} \quad (\lambda - 1)x - 2y = 0 \quad (i)$$

$$3x + 2y = \lambda y \quad \text{or} \quad -3x + (\lambda - 2)y = 0 \quad (ii)$$

which has a solution subject to

$$\begin{bmatrix} (\lambda - 1) & -2 \\ -3 & (\lambda - 2) \end{bmatrix} = (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = -1, \lambda = 4$$

when  $\lambda = 4$  we use (i),(ii) then  $3x - 2y = 0$  then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ is the first eigen vector corresponding the eigen}$$

$$\text{value } \lambda = 4 \text{ and when } \lambda = -1, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

---

*Dr. Fathi Abdallah*